Mathematics 24 Midterm 2 Take-Home Part
Spring 2013
Due Tuesday, May 21 at 2 pm in Room 207 Kemeny
(If no one is there slip the exam under the door.)

This examination consists of four problems. You are to do your own work and not discuss the exam with anyone. For sources you may use the textbook, your homework or your class notes. You may cite a result without proof if it appears in either (1) the assigned reading in the text (2) the assigned homework (3) your class notes. If you cannot solve one part of a problem, you may still use that part in later parts. If you cannot completely solve a problem, you should indicate how far you have gotten.

Important: Write on one side of the paper and show your work. Messy and barely legible papers will not be considered. Give reasons, but try to keep your solutions short.

1. (15 points) Let W be the set of all infinite sequences $(a_1, a_2, \ldots, a_n, \ldots)$, where $a_i \in F$. (W should not be confused with the vector space in Example 5, p. 11). Then W is a vector space with addition and scalar multiplication defined as follows

$$(a_1, a_2, \ldots, a_n, \ldots) + (b_1, b_2, \ldots, b_n, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots)$$

$$a(a_1, a_2, \ldots, a_n, \ldots) = (aa_1, aa_2, \ldots, aa_n, \ldots).$$

You may assume this without proof. Let $V = P(F)$. Prove that V is isomorphic to a subspace of W. (Find the subspace, prove it is a subspace, and show there is a function $T : V \to W$ that gives rise to an isomorphism of V with the subspace.)

2. (15 points) Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a 2×2 matrix with $a \neq 0$ and determinant Δ non-zero. Show that A is a product of at least four elementary matrices. Write down the elementary matrices in order. (Hint: The inverse of an elementary matrix is an elementary matrix.)

3. (12 points) For all $k \geq 1$, represent the elements of F^k as k-dimensional row vectors. Let A be an $m \times n$ and define $\rho_A : F^m \to F^n$, right multiplication by A, by

$$\rho_A(x) = xA, \text{ for every } x \in F^m.$$

1. Show that ρ_A is a linear transformation.
2. What is the matrix of ρ_A with respect to the standard bases?
3. Prove $\operatorname{rank} \rho_A = \operatorname{rank}(A)$.

4. (18 points) Let $T : M_{n\times n}(F) \to M_{n\times n}(F)$ be the linear transformation defined by $T(A) = A^t$.

1. Show that T has two eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = -1$.
2. Let E_{λ_i} be the eigenspace of λ_i, $i = 1, 2$ (Definition p. 264). Prove that $M_{n\times n}(F) = E_{\lambda_1} \oplus E_{\lambda_2}$.
3. In the case $n = 2$, find $\dim E_{\lambda_1}$ and $\dim E_{\lambda_2}$.